DIAS STP 89-21

The Extension Of Chiral Gravity To SL(2, C)

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ABSTRACT

The canonical transformation of the phase space of general relativity introduced by Ashtekar is extended from an $su(2)$ gauge field to $sl(2, C)$. The momenta conjugate to the chiral spin connection become complex, holomorphic variables. It is shown how the symplectic structure of Ashtekar's variables is preserved under Lorentz boosts.

The new canonical variables recently found by Ashtekar, [1,2], have provided a fresh insight into the structure of the constraints in the Hamiltonian formalism of general relativity. In Ashtekar's formalism the phase space of general relativity becomes that of an $su(2)$ Yang-Mills theory, though the Hamiltonian is weakly vanishing and the dynamics are represented purely by constraints. The canonical variables are constructed by foliating four dimensional space-time $\mathcal{M}$ into $\mathbf{R} \times \Sigma$, a direct product of time with a space-like, three dimensional hypersurface Σ. The new phase space co-ordinates are then taken to be the four dimensional self-dual spin connection one forms of $\mathcal{M}$

$$A^i = \omega^{oi} + \frac{i}{2}\epsilon^{ijk}\omega^{jk} \tag{1}$$

restricted to the tangent space of Σ, $A^i{}_a$. The new momenta are the densitised inverse orthonormal triad, tangent to Σ,

$$E^{ai} = det(h^j{}_b)h^{ai} \tag{2}$$

where $h^i{}_a$ are an orthonormal triad on Σ. $a, b, \ldots = 1, 2, 3$ are co-ordinate indices on Σ and $i, j, \ldots = 1, 2, 3$ are orthonormal indices on Σ. $A^i{}_0$ play the role of Lagrange multipliers for Gauss' law, [3]. The Hamiltonian of general relativity in these variables then becomes a linear combination of constraints with Lagrange multipliers, [3,4]. The constraints can be written as, [1][3],

$$\begin{aligned} Tr(\underline{E} \times \underline{E}).\underline{B} &= 0 \\ Tr(\underline{E}.\underline{B}) &= 0 \\ (\underline{D}.\underline{E})^i &= 0 \end{aligned} \tag{3}$$

where the trace is over $su(2)$ and $\underline{B}$ is the $su(2)$ magnetic field on Σ obtained from the four dimensional field strength $F = dA + A \wedge A$. D is the $su(2)$ covariant derivative. These first class constraints represent general co-ordinate invariance and invariance under $su(2)$ tangent space rotations.

The momenta, E^{ai}, are defined in terms of an orthonormal triad on Σ, the fourth member of the tetrad being a unit normal to Σ. The purpose of this paper is to show that this is not necessary, and it is possible to define momenta E^{ai} using an arbitrary tetrad $e^m{}_\mu$ on $\mathcal{M}$, thus extending Ashtekar's $su(2)$ to the full $sl(2, C)$ of the Lorentz group (four dimensional co-ordinate and orthonormal indices will be represented by $\mu, \nu, \ldots$ and $m, n, \ldots$ respectively). E^{ai} will become complex, holomorphic co-ordinates on phase space, thus removing the "generalised Hermitian" constraint which is usually imposed on the densitised triad, [1][3]. The fact that a gauge other than the time gauge removes this Hermitian restriction was first noted in [5]. The restriction to real E^{ai} is a complication in Ashtekar's formalism, the difficulties of which were stressed in [6]. Relaxing this condition to allow E^{ai} to become complex, holomorphic functions restores the symmetry between E^{ai} and $A^i{}_a$.

The transformation properties of the new phase space variables under $sl(2, C)$ will be derived (they are simply complex extensions of the $su(2)$ rotations). It will be shown that this extension to $sl(2, C)$ preserves the symplectic struture on phase space.

We start with the Einstein Lagrangian density

$$\mathcal{L}(x,t) = \frac{1}{2} R(\omega)_{mn} \wedge *e^{mn} \tag{4}$$

where $e^{mn} = e^m{}_\mu e^n{}_\nu dx^\mu \wedge dx^\nu$, and $R_{mn}(\omega)$ are the curvature two forms. We use first order formalism where $e^m{}_\mu$ and $\omega_\mu{}^{mn}$ are considered to be independent of one another. Defining self-dual and anti-self-dual connections by

$$\left(\omega^\pm\right)^{mn} = \frac{1}{2}\left(\pm\omega^{mn} - \frac{i}{2}\epsilon^{mn}{}_{pq}\omega^{pq}\right) \tag{5}$$

($\epsilon_{0123} = 1$, $\eta_{mn} = (-+++)$), the Lagrangian density can be written as

$$\mathcal{L}(x,t) = \frac{1}{2}\left(R(\omega^+)_{mn} - R(\omega^-)_{mn}\right) \wedge *e^{mn}. \tag{6}$$

Following [3] we add the combination

$$\frac{1}{2}\left(R(\omega^+)_{mn} + R(\omega^-)_{mn}\right) \wedge *e^{mn} = \frac{i}{2} R(\omega)_{mn} \wedge e^{mn} \tag{7}$$

to $\mathcal{L}$. This contains the totally antisymmetrised Riemann tensor and so vanishes in the absence of torsion. Even when torsion is present, this term still vanishes under fairly mild conditions on the torsion, [7], which are satisfied, for example, by the torsion induced by spinors, [8]. With this extra term, the Lagrangian density is

$$\mathcal{L}'(x,t) = R(\omega^+)_{mn} \wedge *e^{mn} = \frac{1}{2} R(\omega^+)_{mn} \wedge \left(*e^{mn} + ie^{mn}\right). \tag{8}$$

It is appropriate to say a few words about the choice of co-ordinates at this point. Σ can be taken to be constant time slices of space-time, but this is not necessary. In a general co-ordinate system, x^μ, Σ can be defined by taking a time function $t(x^\mu)$ with Σ being surfaces of constant t with unit normal n_μ. Tensors on Σ are then obtained from tensors on $\mathcal{M}$ by projecting down with the matrix $\delta^\mu{}_\nu + n^\mu n_\nu$, [9]. Time evolution is represented by Lie differentiation with respect to the vector $t_\mu = \partial t/\partial x^\mu$. However this is an unnecessary complication and we shall choose $t = x^0$ without loss of generality. In this co-ordinate system, adapted to Σ, Lie differentiation with respect to t_μ becomes ordinary differentiantion with respect to time. It must be stressed that this is *not* a gauge choice, general co-ordinate invariance is still ensured by the presence of the lapse function and shift vectors below. Of course all this has nothing to do with the choice of tetrad which can still be taken in an arbitrary orientation with respect to Σ.

Now a general orthonormal tetrad can be obtained from one adapted to Σ as follows. Let

$$h^m{}_\mu = \begin{pmatrix} N & 0 \\ h^i{}_c N^c & h^i{}_b \end{pmatrix} \tag{9}$$

be a tetrad with $h^0 = N dx^0$ normal to Σ (this choice of tetrad is usually called the time gauge). Here N is the lapse function and N^a the shift vector of the co-ordinate system x^μ with respect to the foliation $\mathbf{R} \times \Sigma$. $h^i{}_a$ is an orthonormal triad on Σ satisfying

$$h^i{}_a h^i{}_b = g_{ab} \qquad\qquad h^{bi} h^i{}_a = \delta^b{}_a \tag{10}$$

where g_{ab} is the three dimensional metric on Σ.

To the author's knowledge all the literature concerning Ashtekar's canonical transformation uses the orthonormal triad tangent to Σ, defining the new momenta in terms of $h^i{}_a$, which transform under $su(2)$ tangent space rotations of Σ. Lorentz boosts are considered by Hennaux et al.,[5], but they still define the variables E^{ai} using the orthonormal triad $h^i{}_a$. It will be shown below that this is not necessary and it is possible to define E^{ai} using a general tetrad which transforms under $sl(2, C)$.

An arbitrary Lorentz boost, tangent to Σ, with three velocity v^i is given by

$$L(v)^m{}_n = \begin{pmatrix} \gamma & -\gamma v^j \\ -\gamma v^i & \delta^{ij} + \frac{\gamma^2}{1+\gamma} v^i v^j \end{pmatrix}. \tag{11}$$

Thus an arbitrary tetrad is of the form

$$e^m{}_\mu = L(v)^m{}_n h^n{}_\mu. \tag{12}$$

In particular

$$e^i{}_a = h^i{}_a + \frac{\gamma^2}{1+\gamma} v^i v^j h^j{}_a \tag{13}$$

are not an orthonormal triad, since $e^i{}_b g^{ab} \neq (e^{-1})^{ai}$.

Returning now to the Lagrangian density, (8), we see that it can be written as

$$\mathcal{L}' = -iF^i \wedge \left(e^{0i} + \frac{i}{2} \epsilon^{ijk} e^{jk} \right) \tag{14}$$

where $F^i = 2R^{0i}({}^+\omega)$. One finds after some algebra, using (9), (11) and (12),

$$e^{0i} + \frac{i}{2}\epsilon^{ijk} e^{jk} = \epsilon^{abc} \left(\frac{i}{2} E^{ci} dx^a \wedge dx^b + iN^a E^{ci} dt \wedge dx^b + \frac{\underset{\sim}{N}}{2} \epsilon^{ijk} E^{jb} E^{kc} dt \wedge dx^a \right) \tag{15}$$

where

$$E^{ai} = det(e^j{}_b) \left(e^{-1}\right)^{ai} - i\epsilon^{abc} e^i{}_b e^j{}_c v^j \tag{16}$$

and $\underset{\sim}{N} = \gamma N / det(e^i{}_a)$. A formula similar to equation (16) appears in [5], (equation (23) of that reference), but $h^i{}_a$ is used instead of $e^i{}_a$ which is clearly not equivalent.

Thus, after integrating by parts, (14) becomes

$$\mathcal{L}' = \left(E^{ai} \dot{A}^i_a + (D_a E^a)^i A^i{}_0 + N^a E^{bi} F^i{}_{ba} - \frac{i}{2} \underset{\sim}{N} \epsilon^{ijk} E^{bj} E^{ck} F^i_{bc} \right) dt \wedge d^3x \tag{17}$$

which shows that $\underset{\sim}{N}, N^a$ and $A^i{}_0$ are Lagrange multipliers for the constraints (3) above and E^{ai} and $A^i{}_a$ are canonically conjugate. (In general one might expect to have to calculate Dirac brackets between E^{ai} and $A^i{}_a$, since in first order formalism there are second class constraints among $e^m{}_\mu$ and $\omega_\mu{}^{mn}$, but by a simple argument in [3] this is unnecessary.)

It is straightforward to verify that the complex E^{ai}, (16), satisfy

$$E^{ai}E^{bi} = det(g_{cd})g^{ab} \qquad E^{ai}E^{bj}g_{ab} = det(g_{cd})\delta^{ij} \tag{18}$$

and thus can be regarded, in a sense, as a complex orthonormal triad density.

The effect of an infinitesimal Lorentz boost on E^{ai} and $A^i{}_a$ is easily calculated. Using

$$\delta L^m{}_n = \begin{pmatrix} 0 & -\delta v^j \\ -\delta v^i & 0 \end{pmatrix} \tag{19}$$

one finds

$$\begin{aligned} \delta_v A^i{}_a &= (D_a \delta v)^i = \partial_a \delta v^i + i\epsilon^{ijk} A^j{}_a \delta v^k \\ \delta_v E^{ai} &= i\epsilon^{ijk} E^{aj} \delta v^k \end{aligned} \tag{20}$$

which is to be compared with effect of an infinitesimal tangent space rotation on Σ, parameterised by $\delta\theta^i$,

$$\begin{aligned} \delta_\theta A^i{}_a &= -i\,(D_a \delta\theta)^i \\ \delta_\theta E^{ai} &= \epsilon^{ijk} E^{aj} \delta\theta^k. \end{aligned} \tag{21}$$

As an extra check that these variables are canonically conjugate it is instructive to prove that Lorentz transformations leave the Poisson brackets unchanged. We already know that, for $v^i = 0$, E^{ai} (equation (2)) is canonically conjugate to $A^i{}_a$

$$\{A^i{}_a, E^{bj}\} = \delta^{ij}\delta_a{}^b \tag{22}$$

where the number of degrees of freedom between the real E^{ai} and the complex $A^i{}_a$ are matched by restricting $A^i{}_a$ to be holomorphic. It is easy to verify, using (20), that an infinitesimal boost leaves the Poisson bracket unchanged as, of course, do infinitesimal rotations. As boosts and rotations form a group, we can simply exponentiate and deduce that finite Lorentz transformations also leave the Poisson bracket invariant. Hence equation (22) must also hold for the complex E^{ai} with $v^i \neq 0$. The degrees of freedom are now matched by demanding that *both* $A^i{}_a$ and E^{ai} are holomorphic.

In conclusion, it has been shown that it is not necessary to match the choice of orthonormal frame to the foliation of space-time in Ashtekar's chiral gravity. The new momenta (16) are now complex, holomorphic the reality condition no longer being necessary. The Yang-Mills fields become $sl(2,C)$ valued, the $sl(2,C)$ gauge transformations are given by (20) and (21).

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